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APPROXIMATE KINEMATICAL RELATIONS IN PLASTICITY

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Abstract—Within the framework of finite plasticity, it is shown how a number of kinematical approximations can be systematically derived from the multiplicative decomposition of the deformation gradient. Particular attention is devoted to two types of approximate theories: (a) those in which elastic deformations are of a different order of magnitude than plastic deformations; and (b) those in which rotations are of a different order of magnitude than strains.

1. INTRODUCTION

In the classical theory of plasticity, in which deformations are assumed to be infinitesimal, the strain and rotation tensors can be additively decomposed into elastic and plastic parts. Correspondingly, in finite plasticity, the deformation gradient can be multiplicatively decomposed into elastic and plastic parts[1–3]. These, in turn, can be written as products of stretches and rotations by means of the polar decomposition theorem. The strain tensors associated with the finite stretch tensors differ from the strain tensors of infinitesimal plasticity by nonlinear terms, and, moreover, do not obey an additive law of decomposition.

The present paper is concerned with situations which are intermediate between those envisaged by infinitesimal plasticity on the one hand, and by finite plasticity on the other. Two types of deformations are of special interest: (a) those in which elastic deformations are of a different order of magnitude than plastic deformations; and (b) those in which rotations are of a different order of magnitude than strains. For such deformations, the following kinds of questions arise: (i) How are the measures of strain and rotation related to those of infinitesimal plasticity? (ii) Is an additive decomposition of strain possible, even if the appropriate measures of strain are nonlinear? Questions of this type will be considered for a variety of approximate theories.

The method utilized herein for the construction of intermediate approximate theories is suggested by recent work of Naghdi and Vongsarnpigoon[4] and Casey and Naghdi[5]. Recall that, in constructing a theory of infinitesimal deformation, one usually adopts as a measure of smallness the magnitude of the displacement gradient (see e.g. Subsection 3.2 of [6]). Both strains and rotations are then necessarily small. However, if only the strains are assumed to be small, then the rotations need not be small, and *vice versa*. The basic idea introduced by Naghdi and Vongsarnpigoon[4] is that the magnitude of rotation can be measured in terms of the magnitude of strain. This leads to a theory of *moderate rotation*, in which the strains are of order ϵ, but the rotations are of order $\epsilon^{1/2}$. Casey and Naghdi[5] considered the complementary theory in which rotations are of order ϵ while strains are of order $\epsilon^{1/2}$. For an elastic material, the latter theory of *moderate strain* was identified in [5] as representing *physically nonlinear elasticity*.

For elastic-plastic materials, one can also consider theories of moderate rotation and of moderate strain: we do this in Section 6 below. But, even more interestingly, in plasticity theory one can suppose that plastic deformations are moderate while elastic deformations remain small, and *vice versa*. Such developments are discussed in Section 5. In constructing the approximate theories of Sections 5 and 6, we start out with the exact expressions contained in Section 2 and proceed to the second-order theory detailed in Section 4. Infinitesimal plasticity is discussed in Section 3 for purposes of comparison. Lagrangian kinematical measures are used in the explicit developments of Sections 2 through 6, and corresponding developments based on Eulerian measures are indicated in Section 7.

Throughout the paper, direct tensor notation is employed. The transpose, determinant, and norm of a second-order tensor $\mathbf{A}$ are denoted by $\mathbf{A}^T$, det $\mathbf{A}$, and $\|\mathbf{A}\|$, respectively, and the inverse of $\mathbf{A}$, if it exists, is denoted by $\mathbf{A}^{-1}$. $\mathbf{I}$ stands for the identity tensor. Additional mathematical background material can be found in [4–6].

2. EXACT EXPRESSIONS

Consider an elastic-plastic body $\mathcal{B}$ moving in three-dimensional space. Choose an arbitrary fixed reference configuration $\boldsymbol{\kappa}_0$ of $\mathcal{B}$, and let $\boldsymbol{\kappa}$ be the present configuration of $\mathcal{B}$ at time t. Denote the deformation gradient relative to the reference configuration by $\mathbf{F}$ (det $\mathbf{F} > 0$). Next, introduce an intermediate stress-free configuration $\bar{\boldsymbol{\kappa}}$, thereby obtaining the multiplicative decomposition

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p, \tag{2.1}$$

with det $\mathbf{F}_e > 0$, det $\mathbf{F}_p > 0$. Locally, $\mathbf{F}_p$ maps the reference configuration into a stress-free configuration $\bar{\boldsymbol{\kappa}}$, and $\mathbf{F}_e$ maps $\bar{\boldsymbol{\kappa}}$ into the present configuration. The factors $\mathbf{F}_e$ and $\mathbf{F}_p$ are interpreted as elastic and plastic parts of $\mathbf{F}$, respectively.

A multiplicative decomposition was utilized by Backman[1], Lee and Liu[2], and Lee[3].† Issues regarding the existence and uniqueness of the decomposition (2.1), and the matter of appropriate invariance requirements for $\mathbf{F}_e$ and $\mathbf{F}_p$ have been addressed in [7–10]. The possibility of accommodating the decomposition (2.1) within the framework of the general theory of plasticity of Green and Naghdi[11, 12] was established in [7].

Performing right polar decompositions on the three tensors in (2.1), we obtain

$$\mathbf{F} = \mathbf{R}\mathbf{M}, \quad \mathbf{F}_e = \mathbf{R}_e\mathbf{M}_e, \quad \mathbf{F}_p = \mathbf{R}_p\mathbf{M}_p, \tag{2.2}$$

in which $\mathbf{R}$, $\mathbf{R}_e$, and $\mathbf{R}_p$ are proper orthogonal tensors, representing rotations, and $\mathbf{M}$, $\mathbf{M}_e$, and $\mathbf{M}_p$ are symmetric positive definite tensors, representing stretches. We also define total, elastic, and plastic Lagrangian strain tensors by

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \mathbf{E}_e = \tfrac{1}{2}(\mathbf{F}_e^T\mathbf{F}_e - \mathbf{I}), \mathbf{E}_p = \tfrac{1}{2}(\mathbf{F}_p^T\mathbf{F}_p - \mathbf{I}). \tag{2.3}$$

From (2.1) and $(2.3)_{1,2,3}$ it follows that

$$\mathbf{E} - \mathbf{E}_p = \mathbf{F}_p^T\mathbf{E}_e\mathbf{F}_p. \tag{2.4}$$

Introduce now the displacement gradient $\mathbf{H}$, and analogous tensors associated with $\mathbf{F}_e$ and $\mathbf{F}_p$:

$$\mathbf{H} = \mathbf{F} - \mathbf{I}, \quad \mathbf{H}_e = \mathbf{F}_e - \mathbf{I}, \quad \mathbf{H}_p = \mathbf{F}_p - \mathbf{I}. \tag{2.5}$$

It is clear from (2.1) and $(2.5)_{1,2,3}$ that

$$\mathbf{H} = \mathbf{H}_e + \mathbf{H}_p + \mathbf{H}_e\mathbf{H}_p. \tag{2.6}$$

Each of the three tensors $\mathbf{H}$, $\mathbf{H}_e$, and $\mathbf{H}_p$ can be additively decomposed into its symmetric and skew-symmetric parts as follows:

$$\mathbf{H} = \mathbf{e} + \mathbf{w}, \quad \mathbf{H}_e = \mathbf{e}_e + \mathbf{w}_e, \quad \mathbf{H}_p = \mathbf{e}_p + \mathbf{w}_p. \tag{2.7}$$

† Backman[1] wrote (2.1) in inverse form, for the purpose of calculating Eulerian strains, but he then proceeded to consider Lagrangian strains. Lee and Liu[2] added the condition that the intermediate configuration be locally stress-free.

with

$$\begin{aligned}
\mathbf{e} &= \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T) = \mathbf{e}^T, & \mathbf{w} &= \tfrac{1}{2}(\mathbf{H} - \mathbf{H}^T) = -\mathbf{w}^T,\\
\mathbf{e}_e &= \tfrac{1}{2}(\mathbf{H}_e + \mathbf{H}_e^T) = \mathbf{e}_e^T, & \mathbf{w}_e &= \tfrac{1}{2}(\mathbf{H}_e - \mathbf{H}_e^T) = -\mathbf{w}_e^T,\\
\mathbf{e}_p &= \tfrac{1}{2}(\mathbf{H}_p + \mathbf{H}_p^T) = \mathbf{e}_p^T, & \mathbf{w}_p &= \tfrac{1}{2}(\mathbf{H} - \mathbf{H}_p^T) = -\mathbf{w}_p^T.
\end{aligned} \tag{2.8}$$

Using all the relations in (2.3), (2.5), (2.7) and (2.8), we obtain

$$\begin{aligned}
\mathbf{E} &= \mathbf{e} + \tfrac{1}{2}\mathbf{H}^T\mathbf{H} = \mathbf{e} + \tfrac{1}{2}(\mathbf{e}^2 + \mathbf{e}\mathbf{w} - \mathbf{w}\mathbf{e} - \mathbf{w}^2),\\
\mathbf{E}_e &= \mathbf{e}_e + \tfrac{1}{2}\mathbf{H}_e^T\mathbf{H}_e = \mathbf{e}_e + \tfrac{1}{2}(\mathbf{e}_e^2 + \mathbf{e}_e\mathbf{w}_e - \mathbf{w}_e\mathbf{e}_e - \mathbf{w}_e^2),\\
\mathbf{E}_p &= \mathbf{e}_p + \tfrac{1}{2}\mathbf{H}_p^T\mathbf{H}_p = \mathbf{e}_p + \tfrac{1}{2}(\mathbf{e}_p^2 + \mathbf{e}_p\mathbf{w}_p - \mathbf{w}_p\mathbf{e}_p - \mathbf{w}_p^2).
\end{aligned} \tag{2.9}$$

Furthermore, with the aid of (2.6), all the equations in (2.8), and $(2.7)_{2,3}$, we find that

$$\begin{aligned}
\mathbf{e} &= \mathbf{e}_e + \mathbf{e}_p + \tfrac{1}{2}\{\mathbf{H}_e\mathbf{H}_p + \mathbf{H}_p^T\mathbf{H}_e^T\}\\
&= \mathbf{e}_e + \mathbf{e}_p + \tfrac{1}{2}\{\mathbf{e}_e\mathbf{e}_p + \mathbf{e}_e\mathbf{w}_p + \mathbf{w}_e\mathbf{e}_p + \mathbf{w}_e\mathbf{w}_p + \mathbf{e}_p\mathbf{e}_e - \mathbf{w}_p\mathbf{e}_e - \mathbf{e}_p\mathbf{w}_e + \mathbf{w}_p\mathbf{w}_e\}
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
\mathbf{w} &= \mathbf{w}_e + \mathbf{w}_p + \tfrac{1}{2}\{\mathbf{H}_e\mathbf{H}_p - \mathbf{H}_p^T\mathbf{H}_e^T\}\\
&= \mathbf{w}_e + \mathbf{w}_p + \tfrac{1}{2}\{\mathbf{e}_e\mathbf{e}_p + \mathbf{e}_e\mathbf{w}_p + \mathbf{w}_e\mathbf{e}_p + \mathbf{w}_e\mathbf{w}_p - \mathbf{e}_p\mathbf{e}_e + \mathbf{w}_p\mathbf{e}_e + \mathbf{e}_p\mathbf{w}_e - \mathbf{w}_p\mathbf{w}_e\}.
\end{aligned} \tag{2.11}$$

The tensors $\mathbf{e}$ and $\mathbf{w}$ are strain and rotation measures appropriate for infinitesimal deformations. However, even for finite deformations $\mathbf{e}$ and $\mathbf{w}$ are unambiguously defined by $(2.8)_{1,2}$. The exact expression in (2.9) which relates $\mathbf{e}$ and $\mathbf{w}$ to the finite strain tensor $\mathbf{E}$ is especially useful in constructing approximate theories in which strain and rotation are allowed to have different orders of magnitude. In this connection, see Section 3 of [5].

3. INFINITESIMAL PLASTICITY

Let a measure ϵ of smallness be defined by

$$\epsilon = \max\{\sup_{\bar{\kappa}} \|\mathbf{H}_e\|, \ \sup_{\kappa_0} \|\mathbf{H}_p\|\} \tag{3.1}$$

(in which sup stands for supremum). If $\mathbf{Z}$ is any tensor-valued function of $(\mathbf{H}_e, \mathbf{H}_p)$ defined in a neighborhood of $(\mathbf{0}, \mathbf{0})$ and satisfying the condition that there exists a nonnegative real constant K such that

$$\|\mathbf{Z}\| < K\epsilon^n \text{ as } \epsilon \to 0, \tag{3.2}$$

where n is a positive number, then we write

$$\mathbf{Z} = O(\epsilon^n). \tag{3.3}$$

In the light of $(2.8)_{3,4,5,6}$ and (3.1),

$$\mathbf{e}_e = O(\epsilon), \quad \mathbf{w}_e = O(\epsilon), \quad \mathbf{e}_p = O(\epsilon), \quad \mathbf{w}_p = O(\epsilon). \tag{3.4}$$

It follows from $(2.2)_{2,3}$, $(2.5)_{2,3}$, $(2.8)_{3,4}$, (3.1), and $(3.4)_{1,3}$ that

$$\mathbf{M}_e = (\mathbf{F}_e^T\mathbf{F}_e)^{1/2} = \mathbf{I} + \mathbf{e}_e + O(\epsilon^2), \quad \mathbf{M}_p = (\mathbf{F}_p^T\mathbf{F}_p)^{1/2} = \mathbf{I} + \mathbf{e}_p + O(\epsilon^2), \tag{3.5}$$

and

$$\mathbf{M}_e^{-1} = \mathbf{I} - \mathbf{e}_e + O(\epsilon^2), \quad \mathbf{M}_p^{-1} = \mathbf{I} - \mathbf{e}_p + O(\epsilon^2). \tag{3.6}$$

Likewise, by virtue of $(2.2)_{2,3}$, $(2.5)_{2,3}$, $(3.6)_{1,2}$, $(2.7)_{2,3}$, and $(3.4)_{1,2,3,4}$,

$$\mathbf{R}_e = \mathbf{F}_e\mathbf{M}_e^{-1} = \mathbf{I} + \mathbf{w}_e + O(\epsilon^2), \quad \mathbf{R}_p = \mathbf{F}_p\mathbf{M}_p^{-1} = \mathbf{I} + \mathbf{w}_p + O(\epsilon^2). \tag{3.7}$$

Also, in view of $(2.2)_{2,3}$, $(3.6)_{1,2}$, $(3.7)_{1,2}$, $(2.8)_{4,6}$, $(2.7)_{2,3}$, and $(3.4)_{1,2,3,4}$,

$$\mathbf{F}_e^{-1} = \mathbf{M}_e^{-1}\mathbf{R}_e^T = \mathbf{I} - \mathbf{H}_e + O(\epsilon^2), \quad \mathbf{F}_p^{-1} = \mathbf{M}_p^{-1}\mathbf{R}_p^T = \mathbf{I} - \mathbf{H}_p + O(\epsilon^2). \tag{3.8}$$

It is clear from $(2.9)_{2,3}$, (3.1), and $(3.4)_{1,3}$ that

$$\mathbf{E}_e = \mathbf{e}_e + O(\epsilon^2) = O(\epsilon), \quad \mathbf{E}_p = \mathbf{e}_p + O(\epsilon^2) = O(\epsilon). \tag{3.9}$$

The relations $(3.9)_{1,2}$ are the justification for calling $\mathbf{e}_e$ and $\mathbf{e}_p$ infinitesimal elastic strain and infinitesimal plastic strain, respectively. The tensors $\mathbf{w}_e$ and $\mathbf{w}_p$ are referred to as infinitesimal elastic and plastic rotations, respectively, and are related to $\mathbf{R}_e$ and $\mathbf{R}_p$ through $(3.7)_{1,2}$.

It follows from (2.6) and (3.1) that

$$\mathbf{H} = \mathbf{H}_e + \mathbf{H}_p + O(\epsilon^2) = O(\epsilon). \tag{3.10}$$

Hence, in view of $(2.5)_1$, $(2.2)_1$, and $(2.8)_{1,2}$,

$$\begin{aligned} \mathbf{M} &= (\mathbf{F}^T\mathbf{F})^{1/2} = \mathbf{I} + \mathbf{e} + O(\epsilon^2), \quad \mathbf{M}^{-1} = \mathbf{I} - \mathbf{e} + O(\epsilon^2), \\ \mathbf{R} &= \mathbf{I} + \mathbf{w} + O(\epsilon^2), \quad \mathbf{F}^{-1} = \mathbf{I} - \mathbf{H} + O(\epsilon^2). \end{aligned} \tag{3.11}$$

Finally, it is evident from $(2.10)_1$, $(2.11)_1$, (3.1), and $(3.4)_{1,3}$ that

$$\mathbf{e} = \mathbf{e}_e + \mathbf{e}_p + O(\epsilon^2) = O(\epsilon), \quad \mathbf{w} = \mathbf{w}_e + \mathbf{w}_p + O(\epsilon^2) = O(\epsilon). \tag{3.12}$$

In the infinitesimal theory, all terms of $O(\epsilon^2)$ are neglected.

4. SECOND-ORDER RELATIONS

In this section, we examine the form which the various kinematical relations take when terms up to $O(\epsilon^2)$ are explicitly retained.

It is obvious that $(3.4)_{1,2,3,4}$ still hold, and that all of the terms in (2.6), $(2.9)_{1,2,3}$, $(2.10)_{1,2}$, and $(2.11)_{1,2}$ must be retained. However, we observe that now

$$\begin{aligned} \mathbf{H} &= O(\epsilon), \quad \mathbf{e} = O(\epsilon), \quad \mathbf{w} = O(\epsilon), \\ \mathbf{E} &= O(\epsilon), \quad \mathbf{E}_e = O(\epsilon), \quad \mathbf{E}_p = O(\epsilon). \end{aligned} \tag{4.1}$$

Following the procedure of Casey and Naghdi[5], we write $\mathbf{M}$ in the form

$$\mathbf{M} = \mathbf{I} + \alpha_0\mathbf{e} + \alpha_1\mathbf{e}^2 + \alpha_2\mathbf{ew} + \alpha_3\mathbf{we} + \alpha_4\mathbf{w}^2 + O(\epsilon^3), \tag{4.2}$$

where the coefficients α_0, α_1, α_2, α_3, and α_4 are to be determined. To do this, we note from $(2.2)_1$, $(2.5)_1$, and $(2.7)_1$ that

$$\mathbf{M}^2 = \mathbf{I} + 2\mathbf{e} + \mathbf{e}^2 + \mathbf{ew} - \mathbf{we} - \mathbf{w}^2. \tag{4.3}$$

Then, substituting (4.2) in (4.3) and solving for the coefficients, we deduce that

$$\mathbf{M} = \mathbf{I} + \mathbf{e} + \tfrac{1}{2}(\mathbf{ew} - \mathbf{we} - \mathbf{w}^2) + O(\epsilon^3). \tag{4.4}$$

Similarly,

$$\begin{aligned} \mathbf{M}_e &= \mathbf{I} + \mathbf{e}_e + \tfrac{1}{2}(\mathbf{e}_e\mathbf{w}_e - \mathbf{w}_e\mathbf{e}_e - \mathbf{w}_e^2) + O(\epsilon^3), \\ \mathbf{M}_p &= \mathbf{I} + \mathbf{e}_p + \tfrac{1}{2}(\mathbf{e}_p\mathbf{w}_p - \mathbf{w}_p\mathbf{e}_p - \mathbf{w}_p^2) + O(\epsilon^3). \end{aligned} \tag{4.5}$$

Using the same procedure, we also obtain

$$\begin{aligned} \mathbf{M}^{-1} &= \mathbf{I} - \mathbf{e} + \tfrac{1}{2}(2\mathbf{e}^2 - \mathbf{ew} + \mathbf{we} + \mathbf{w}^2) + O(\epsilon^3), \\ \mathbf{M}_e^{-1} &= \mathbf{I} - \mathbf{e}_e + \tfrac{1}{2}(2\mathbf{e}_e^2 - \mathbf{e}_e\mathbf{w}_e + \mathbf{w}_e\mathbf{e}_e + \mathbf{w}_e^2) + O(\epsilon^3), \\ \mathbf{M}_p^{-1} &= \mathbf{I} - \mathbf{e}_p + \tfrac{1}{2}(2\mathbf{e}_p^2 - \mathbf{e}_p\mathbf{w}_p + \mathbf{w}_p\mathbf{e}_p + \mathbf{w}_p^2) + O(\epsilon^3). \end{aligned} \tag{4.6}$$

It follows from $(4.6)_{1,2,3}$, $(2.2)_{1,2,3}$, $(2.5)_{1,2,3}$, and $(2.7)_{1,2,3}$ that

$$\begin{aligned} \mathbf{R} &= \mathbf{I} + \mathbf{w} - \tfrac{1}{2}(\mathbf{ew} + \mathbf{we} - \mathbf{w}^2) + O(\epsilon^3), \\ \mathbf{R}_e &= \mathbf{I} + \mathbf{w}_e - \tfrac{1}{2}(\mathbf{e}_e\mathbf{w}_e + \mathbf{w}_e\mathbf{e}_e - \mathbf{w}_e^2) + O(\epsilon^3), \\ \mathbf{R}_p &= \mathbf{I} + \mathbf{w}_p - \tfrac{1}{2}(\mathbf{e}_p\mathbf{w}_p + \mathbf{w}_p\mathbf{e}_p - \mathbf{w}_p^2) + O(\epsilon^3). \end{aligned} \tag{4.7}$$

Furthermore, proceeding as in (4.2), we find that

$$\begin{aligned} \mathbf{F}^{-1} &= \mathbf{I} - \mathbf{H} + \mathbf{H}^2 + O(\epsilon^3), \\ \mathbf{F}_e^{-1} &= \mathbf{I} - \mathbf{H}_e + \mathbf{H}_e^2 + O(\epsilon^3), \\ \mathbf{F}_p^{-1} &= \mathbf{I} - \mathbf{H}_p + \mathbf{H}_p^2 + O(\epsilon^3). \end{aligned} \tag{4.8}$$

The latter set of expressions may also be deduced from $(4.6)_{1,2,3}$, and $(4.7)_{1,2,3}$ in the same manner as $(3.8)_{1,2}$ were established.

Finally, with the use of (2.4), $(2.5)_3$, $(2.9)_2$, (3.1), $(3.4)_{1,2,3,4}$, and $(4.1)_5$, we obtain

$$\begin{aligned} \mathbf{E} - \mathbf{E}_p &= \mathbf{E}_e + \mathbf{H}_p^T\mathbf{e}_e + \mathbf{e}_e\mathbf{H}_p + O(\epsilon^3) \\ &= \mathbf{E}_e + (\mathbf{e}_p - \mathbf{w}_e)\mathbf{e}_e + \mathbf{e}_e(\mathbf{e}_p + \mathbf{w}_p) + O(\epsilon^3) = O(\epsilon). \end{aligned} \tag{4.9}$$

In the second-order theory, all terms of $O(\epsilon^3)$ are neglected.

The expressions (4.4), $(4.6)_1$, $(4.7)_1$ and $(4.8)_1$ were given by Casey and Naghdi[5]. The relationship between $\mathbf{E}$, $\mathbf{E}_e$, $\mathbf{E}_p$, $\mathbf{e}_e$, $\mathbf{e}_p$, $\mathbf{w}_e$, and $\mathbf{w}_p$ indicated in (4.9) is similar to one obtained by Backman†. However, Backman retains some terms of $O(\epsilon^2)$ in his expression, while neglecting others of the same order of magnitude.

5. ELASTIC DEFORMATIONS OF DIFFERENT ORDER OF MAGNITUDE THAN PLASTIC DEFORMATIONS

In this section, we shall allow $\mathbf{H}_e$ and $\mathbf{H}_p$ to have different orders of magnitude. Two complementary cases will be considered: first, the case of small plastic deformations accompanied by moderate elastic deformations, and second, the case of small elastic deformations accompanied by moderate plastic deformations.

† See equation (16) of [1].

5.1 *Small plastic deformations, moderate elastic deformations*
In this subsection, we let

$$\bar{\epsilon} = \sup_{\bar{\kappa}_0} \| \mathbf{H}_p \|. \tag{5.1}$$

We say that $\mathbf{F}_e$ corresponds to a *moderate elastic deformation* with respect to $\bar{\epsilon}$ if

$$\mathbf{H}_e = O(\bar{\epsilon}^{1/2}). \tag{5.2}$$

It is obvious from $(2.8)_{3,4,5,6}$, (5.1), and (5.2) that now

$$\mathbf{e}_p = O(\bar{\epsilon}), \quad \mathbf{w}_p = O(\bar{\epsilon}), \quad \mathbf{e}_e = O(\bar{\epsilon}^{1/2}), \quad \mathbf{w}_e = O(\bar{\epsilon}^{1/2}). \tag{5.3}$$

Furthermore, it follows from (2.6), $(2.8)_{1,2}$, (5.1), and (5.2) that

$$\mathbf{H} = \mathbf{H}_e + \mathbf{H}_p + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}), \quad \mathbf{e} = O(\bar{\epsilon}^{1/2}), \quad \mathbf{w} = O(\bar{\epsilon}^{1/2}). \tag{5.4}$$

In $(2.9)_{1,2}$, all terms must be retained, but in $(2.9)_3$, only $\mathbf{e}_p$ is of order greater than $O(\bar{\epsilon}^{1/2})$. Thus,

$$\begin{aligned}
\mathbf{E} &= \mathbf{e} + \tfrac{1}{2}(\mathbf{e}^2 + \mathbf{e}\mathbf{w} - \mathbf{w}\mathbf{e} - \mathbf{w}^2) = O(\bar{\epsilon}^{1/2}) \\
\mathbf{E}_e &= \mathbf{e}_e + \tfrac{1}{2}(\mathbf{e}_e^2 + \mathbf{e}_e\mathbf{w}_e - \mathbf{w}_e\mathbf{e}_e - \mathbf{w}_e^2) = O(\bar{\epsilon}^{1/2}) \\
\mathbf{E}_p &= \mathbf{e}_p + O(\bar{\epsilon}^2) = \mathbf{e}_p + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}),
\end{aligned} \tag{5.5}$$

where use has been made of $(5.4)_{2,3}$ and $(5.3)_{1,2,3,4}$. With the aid of $(5.3)_{1,2,3,4}$, the relations (2.10) and (2.11) reduce to

$$\mathbf{e} = \mathbf{e}_e + \mathbf{e}_p + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}), \quad \mathbf{w} = \mathbf{w}_e + \mathbf{w}_p + O(\bar{\epsilon}^{3/2}) = (\bar{\epsilon}^{1/2}). \tag{5.6}$$

The following expressions for the stretch tensors may be deduced from (4.4) and $(4.5)_{1,2}$ with the use of $(5.4)_{2,3}$, $(5.3)_{1,2,3,4}$, (3.1), and (5.1):

$$\begin{aligned}
\mathbf{M} &= \mathbf{I} + \mathbf{e} + \tfrac{1}{2}(\mathbf{e}\mathbf{w} - \mathbf{w}\mathbf{e} - \mathbf{w}^2) + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}) \\
\mathbf{M}_e &= \mathbf{I} + \mathbf{e}_e + \tfrac{1}{2}(\mathbf{e}_e\mathbf{w}_e - \mathbf{w}_e\mathbf{e}_e - \mathbf{w}_e^2) + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\
\mathbf{M}_p &= \mathbf{I} + \mathbf{e}_p + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}).
\end{aligned} \tag{5.7}$$

Similarly, from $(4.6)_{1,2,3}$ one obtains

$$\begin{aligned}
\mathbf{M}^{-1} &= \mathbf{I} - \mathbf{e} + \tfrac{1}{2}(2\mathbf{e}^2 - \mathbf{e}\mathbf{w} + \mathbf{w}\mathbf{e} + \mathbf{w}^2) + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\
\mathbf{M}_e^{-1} &= \mathbf{I} - \mathbf{e}_e + \tfrac{1}{2}(2\mathbf{e}_e^2 - \mathbf{e}_e\mathbf{w}_e + \mathbf{w}_e\mathbf{e}_e + \mathbf{w}_e^2) + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\
\mathbf{M}_p^{-1} &= \mathbf{I} - \mathbf{e}_p - O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}).
\end{aligned} \tag{5.8}$$

Likewise, $(4.7)_{1,2,3}$ lead to the following expressions for the rotation tensors:

$$\begin{aligned}
\mathbf{R} &= \mathbf{I} + \mathbf{w} - \tfrac{1}{2}(\mathbf{e}\mathbf{w} + \mathbf{w}\mathbf{e} - \mathbf{w}^2) + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}) \\
\mathbf{R}_e &= \mathbf{I} + \mathbf{w}_e - \tfrac{1}{2}(\mathbf{e}_e\mathbf{w}_e + \mathbf{w}_e\mathbf{e}_e - \mathbf{w}_e^2) + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}) \\
\mathbf{R}_p &= \mathbf{I} + \mathbf{w}_p + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}).
\end{aligned} \tag{5.9}$$

Also, by virtue of $(5.4)_1$, (5.2), (5.1), and (3.1), it is evident from $(4.8)_{1,2,3}$ that

$$\begin{aligned} \mathbf{F}^{-1} &= \mathbf{I} - \mathbf{H} + \mathbf{H}^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{F}_e^{-1} &= \mathbf{I} - \mathbf{H}_e + \mathbf{H}_e^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{F}_p^{-1} &= \mathbf{I} - \mathbf{H}_p + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}). \end{aligned} \tag{5.10}$$

Moreover, with the aid of (5.1), $(5.3)_3$, $(5.5)_3$ and (3.1), we deduce from (4.9) that

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_e + \mathbf{E}_p + O(\bar{\epsilon}^{3/2}) \\ &= \mathbf{E}_e + \mathbf{e}_p + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{5.11}$$

An approximate theory may be based on the above results by neglecting all terms of $O(\bar{\epsilon}^{3/2})$. In this theory, additive decompositions hold in the forms $(5.6)_{1,2}$ and (5.11). Also, as can be seen from $(5.5)_3$ and $(5.9)_3$, $\mathbf{e}_p$ and $\mathbf{w}_p$ are appropriate measures of plastic strain and plastic rotation.

5.2 *Small elastic deformations, moderate plastic deformations*

Instead of (5.1), in this subsection we let

$$\bar{\epsilon} = \sup_{\bar{\kappa}} \| \mathbf{H}_e \|. \tag{5.12}$$

We say that $\mathbf{F}_p$ corresponds to a *moderate plastic deformation* with respect to $\bar{\epsilon}$ in (5.12) if

$$\mathbf{H}_p = O(\bar{\epsilon}^{1/2}). \tag{5.13}$$

Expressions complementary to those of Subsection 5.1 may now be readily deduced by the same type of arguments as were used in that subsection. In place of (5.3), we now have

$$\mathbf{e}_e = O(\bar{\epsilon}), \quad \mathbf{w}_e = O(\bar{\epsilon}), \quad \mathbf{e}_p = O(\bar{\epsilon}^{1/2}), \quad \mathbf{w}_p = O(\bar{\epsilon}^{1/2}). \tag{5.14}$$

The expressions $(5.4)_{1,2,3}$ hold in the same form (with $\bar{\epsilon}$ being now given by (5.12)). The expression for $\mathbf{E}$ is of the same form as in $(5.5)_1$, while

$$\begin{aligned} \mathbf{E}_e &= \mathbf{e}_e + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}), \\ \mathbf{E}_p &= \mathbf{e}_p + \tfrac{1}{2}(\mathbf{e}_p^2 + \mathbf{e}_p\mathbf{w}_p - \mathbf{w}_p\mathbf{e}_p - \mathbf{w}_p^2) = O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{5.15}$$

The form of the results $(5.6)_{1,2}$ remains unchanged.

The expressions for $\mathbf{M}$, $\mathbf{M}^{-1}$, $\mathbf{R}$, $\mathbf{F}^{-1}$ have the same form as in $(5.7)_1$, $(5.8)_1$, $(5.9)_1$, and $(5.10)_1$, respectively, while

$$\begin{aligned} \mathbf{M}_e &= \mathbf{I} + \mathbf{e}_e + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{M}_e^{-1} &= \mathbf{I} - \mathbf{e}_e + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{M}_p &= \mathbf{I} + \mathbf{e}_p + \tfrac{1}{2}(\mathbf{e}_p\mathbf{w}_p - \mathbf{w}_p\mathbf{e}_p - \mathbf{w}_p^2) + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{M}_p^{-1} &= \mathbf{I} - \mathbf{e}_p + \tfrac{1}{2}(2\mathbf{e}_p^2 - \mathbf{e}_p\mathbf{w}_p + \mathbf{w}_p\mathbf{e}_p + \mathbf{w}_p^2) + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{R}_e &= \mathbf{I} + \mathbf{w}_e + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{R}_p &= \mathbf{I} + \mathbf{w}_p - \tfrac{1}{2}(\mathbf{e}_p\mathbf{w}_p + \mathbf{w}_p\mathbf{e}_p - \mathbf{w}_p^2) + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{F}_e^{-1} &= \mathbf{I} - \mathbf{H}_e + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{F}_p^{-1} &= \mathbf{I} - \mathbf{H}_p + \mathbf{H}_p^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{5.16}$$

Also, paralleling (5.11), it now follows from (4.9) that

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_e + \mathbf{E}_p + O(\bar{\epsilon}^{3/2}) \\ &= \mathbf{e}_e + \mathbf{E}_p + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{5.17}$$

All terms of $O(\bar{\epsilon}^{3/2})$ are omitted in the aproximate theory. In this theory, additive decompositions hold in the form $(5.6)_{1,2}$ and (5.17). The tensors $\mathbf{e}_e$ and $\mathbf{w}_e$ are appropriate measures of elastic strain and elastic rotation.

6. ROTATIONS OF DIFFERENT ORDER OF MAGNITUDE THAN STRAINS

In this section, we consider another pair of complementary cases, namely the case of small strains accompanied by moderate rotations and the case of small rotations accompanied by moderate strains.

The possibility of measuring rotation in terms of the magnitude of strain was first recognized by Naghdi and Vongsarnpigoon[4]. These authors gave a precise definition of moderate rotation and also identified sufficiency conditions for the rotation to be moderate (see Definition 4.2 and Theorem 4.4 of [4]). An alternative but equivalent definition of moderate rotation was proposed by Casey and Naghdi[5]. The latter definition will be used in Subsection 6.1. In Subsection 6.2, again following Casey and Naghdi[5], we consider small rotations accompanied by moderate strains. In the case of an elastic material, this leads to a theory of physically nonlinear elasticity (see Subsection 3.4 of [5]).

6.1 *Small strains, moderate rotations*

In the present subsection, we set

$$\bar{\epsilon} = \max\{\sup_{\bar{\kappa}} \| \mathbf{e}_e \|, \quad \sup_{\kappa_0} \| \mathbf{e}_p \|\}. \tag{6.1}$$

We say that $\mathbf{w}_e$ and $\mathbf{w}_p$ correspond to *moderate rotations* with respect to $\bar{\epsilon}$ if

$$\mathbf{w}_e = O(\bar{\epsilon}^{1/2}), \quad \mathbf{w}_p = O(\bar{\epsilon}^{1/2}). \tag{6.2}$$

It is clear from $(2.7)_{2,3}$, (2.6), (6.1), and (6.2) that

$$\begin{aligned} \mathbf{H}_e &= O(\bar{\epsilon}^{1/2}), \quad \mathbf{H}_p = O(\bar{\epsilon}^{1/2}), \\ \mathbf{H} &= \mathbf{H}_e + \mathbf{H}_p + \mathbf{H}_e\mathbf{H}_p = O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{6.3}$$

It follows from the second equations in (2.10) and (2.11), (6.1), and $(6.2)_{1,2}$ that

$$\begin{aligned} \mathbf{e} &= \mathbf{e}_e + \mathbf{e}_p + \tfrac{1}{2}(\mathbf{w}_e\mathbf{w}_p + \mathbf{w}_p\mathbf{w}_e) + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}), \\ \mathbf{w} &= \mathbf{w}_e + \mathbf{w}_p + \tfrac{1}{2}(\mathbf{w}_e\mathbf{w}_p - \mathbf{w}_p\mathbf{w}_e) + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{6.4}$$

With the aid of $(6.4)_{1,2}$, $(6.2)_{1,2}$, and (6.1), the finite strain tensors in $(2.9)_{1,2,3}$ become

$$\begin{aligned} \mathbf{E} &= \mathbf{e} - \tfrac{1}{2}\mathbf{w}^2 + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}), \\ \mathbf{E}_e &= \mathbf{e}_e - \tfrac{1}{2}\mathbf{w}_e^2 + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}), \\ \mathbf{E}_p &= \mathbf{e}_p - \tfrac{1}{2}\mathbf{w}_p^2 + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}). \end{aligned} \tag{6.5}$$

In view of (6.1), $(6.2)_{1,2}$, $(6.4)_{1,2}$, and (3.1), the stretch tensors in (4.4) and $(4.5)_{1,2}$ may be written as

$$\begin{aligned} \mathbf{M} &= \mathbf{I} + \mathbf{e} - \tfrac{1}{2}\mathbf{w}^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{M}_e &= \mathbf{I} + \mathbf{e}_e - \tfrac{1}{2}\mathbf{w}_e^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{M}_p &= \mathbf{I} + \mathbf{e}_p - \tfrac{1}{2}\mathbf{w}_p^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}). \end{aligned} \tag{6.6}$$

Similarly, $(4.6)_{1,2,3}$ and $(4.7)_{1,2,3}$ furnish the expressions

$$\begin{aligned} \mathbf{M}^{-1} &= \mathbf{I} - \mathbf{e} + \tfrac{1}{2}\mathbf{w}^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{M}_e^{-1} &= \mathbf{I} - \mathbf{e}_e + \tfrac{1}{2}\mathbf{w}_e^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{M}_p^{-1} &= \mathbf{I} - \mathbf{e}_p + \tfrac{1}{2}\mathbf{w}_p^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} \mathbf{R} &= \mathbf{I} + \mathbf{w} + \tfrac{1}{2}\mathbf{w}^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{R}_e &= \mathbf{I} + \mathbf{w}_e + \tfrac{1}{2}\mathbf{w}_e^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{R}_p &= \mathbf{I} + \mathbf{w}_p + \tfrac{1}{2}\mathbf{w}_p^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{6.8}$$

With the help of $(2.7)_{1,2,3}$, (6.1), $(6.2)_{1,2}$, $(6.4)_{1,2}$, and (3.1), we deduce from $(4.8)_{1,2,3}$ that

$$\begin{aligned} \mathbf{F}^{-1} &= \mathbf{I} - \mathbf{e} - \mathbf{w} + \mathbf{w}^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{F}_e^{-1} &= \mathbf{I} - \mathbf{e}_e - \mathbf{w}_e + \mathbf{w}_e^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}) \\ \mathbf{F}_p^{-1} &= \mathbf{I} - \mathbf{e}_p - \mathbf{w}_p + \mathbf{w}_p^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{6.9}$$

It follows from (4.9), (6.1), $(6.2)_{1,2}$, (3.1) and $(6.5)_2$ that

$$\mathbf{E} - \mathbf{E}^p = \mathbf{E}_e + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}), \tag{6.10}$$

and hence, by virtue of $(6.5)_{1,2,3}$,

$$\mathbf{e} = \mathbf{e}_e + \mathbf{e}_p + \tfrac{1}{2}(\mathbf{w}^2 - \mathbf{w}_e^2 - \mathbf{w}_p^2) + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}). \tag{6.11}$$

If $(6.4)_2$ is substituted in (6.11), $(6.4)_1$ will result.

Once terms of order $\bar{\epsilon}^{3/2}$ are neglected, an additive decomposition is obeyed by the tensors $\mathbf{E}$, $\mathbf{E}_e$ and $\mathbf{E}_p$, as can be seen from (6.10). Also, by $(6.5)_{1,2,3}$, each of these tensors differs from the corresponding infinitesimal strain tensor by a quadratic term in infinitesimal rotation.

6.2 *Small rotations, moderate strains*

In this subsection, we set

$$\bar{\epsilon} = \max\{\sup_{\bar{\kappa}} \|\mathbf{w}_e\|, \ \sup_{\kappa_0} \|\mathbf{w}_p\|\}. \tag{6.12}$$

We then say that $\mathbf{e}_e$ and $\mathbf{e}_p$ correspond to *moderate strains* with respect to $\bar{\epsilon}$ if

$$\mathbf{e}_e = O(\bar{\epsilon}^{1/2}), \quad \mathbf{e}_p = O(\bar{\epsilon}^{1/2}). \tag{6.13}$$

Clearly, the expressions $(6.3)_{1,2,3}$ again hold with $\bar{\epsilon}$ now given by (6.12). Returning to (2.10) and (2.11), and invoking (6.12), $(6.13)_{1,2}$, and (3.1), we deduce that

$$\begin{aligned} \mathbf{e} &= \mathbf{e}_e + \mathbf{e}_p + \tfrac{1}{2}(\mathbf{e}_e\mathbf{e}_p + \mathbf{e}_p\mathbf{e}_e) + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}), \\ \mathbf{w} &= \mathbf{w}_e + \mathbf{w}_p + \tfrac{1}{2}(\mathbf{e}_e\mathbf{e}_p - \mathbf{e}_p\mathbf{e}_e) + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{6.14}$$

Utilizing $(2.9)_{1,2,3}$, $(6.14)_{1,2}$, $(6.13)_{1,2}$, sand (6.12), we obtain

$$\begin{aligned} \mathbf{E} &= \mathbf{e} + \tfrac{1}{2}\mathbf{e}^2 + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}), \\ \mathbf{E}_e &= \mathbf{e}_e + \tfrac{1}{2}\mathbf{e}_e^2 + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}), \\ \mathbf{E}_p &= \mathbf{e}_p + \tfrac{1}{2}\mathbf{e}_p^2 + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{6.15}$$

With the use of (4.4), $(4.5)_{1,2}$, $(4.6)_{1,2,3}$, $(4.7)_{1,2,3}$, $(6.14)_{1,2}$, $(6.13)_{1,2}$, (6.12) and (3.1), we find that

$$\begin{aligned} \mathbf{M} &= \mathbf{I} + \mathbf{e} + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{M}_e &= \mathbf{I} + \mathbf{e}_e + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{M}_p &= \mathbf{I} + \mathbf{e}_p + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \end{aligned} \tag{6.16}$$

while

$$\begin{aligned} \mathbf{M}^{-1} &= \mathbf{I} - \mathbf{e} + \mathbf{e}^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{M}_e^{-1} &= \mathbf{I} - \mathbf{e}_e + \mathbf{e}_e^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{M}_p^{-1} &= \mathbf{I} - \mathbf{e}_p + \mathbf{e}_p^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \end{aligned} \tag{6.17}$$

and

$$\begin{aligned} \mathbf{R} &= \mathbf{I} + \mathbf{w} + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{R}_e &= \mathbf{I} + \mathbf{w}_e + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}), \\ \mathbf{R}_p &= \mathbf{I} + \mathbf{w}_p + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}). \end{aligned} \tag{6.18}$$

From $(4.8)_{1,2,3}$ $(2.7)_{1,2,3}$, $(6.14)_{1,2}$, $(6.13)_{1,2}$, (6.12) and (3.1), we obtain

$$\begin{aligned} \mathbf{F}^{-1} &= \mathbf{I} - \mathbf{e} - \mathbf{w} + \mathbf{e}^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{F}_e^{-1} &= \mathbf{I} - \mathbf{e}_e - \mathbf{w}_e + \mathbf{e}_e^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}), \\ \mathbf{F}_p^{-1} &= \mathbf{I} - \mathbf{e}_p - \mathbf{w}_p + \mathbf{e}_p^2 + O(\bar{\epsilon}^{3/2}) = \mathbf{I} + O(\bar{\epsilon}^{1/2}). \end{aligned} \tag{6.19}$$

Also, it follows froom (4.9), (6.12), $(6.13)_{1,2}$ and $(6.15)_2$ that

$$\mathbf{E} - \mathbf{E}_p = \mathbf{E}_e + \mathbf{e}_e\mathbf{e}_p + \mathbf{e}_p\mathbf{e}_e + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}) \tag{6.20}$$

and hence, by $(6.15)_{1,2,3}$, that

$$\mathbf{e} - \mathbf{e}_p = \mathbf{e}_e - \tfrac{1}{2}(\mathbf{e}^2 - \mathbf{e}_e^2 - \mathbf{e}_p^2) + \mathbf{e}_e\mathbf{e}_p + \mathbf{e}_p\mathbf{e}_e + O(\bar{\epsilon}^{3/2}) = O(\bar{\epsilon}^{1/2}). \tag{6.21}$$

The substitution of $(6.14)_1$ in (6.21) reduces the latter equation to $(6.14)_1$.

Even when terms of order $\bar{\epsilon}^{3/2}$ are neglected in the present subsection, it is clear from (6.20) and (6.21) that additive decompositions of strain do not hold. It is worth noting, however, that the infinitesimal strain tensors differ from the Lagrangian strain tensors only by a quadratic term in infinitesimal strain (see $(6.15)_{1,2,3}$).

Besides the developments described above, several other similar approximate theories may be constructed. For example, one could take $\mathbf{e}_p$ of $O(\bar{\epsilon}^{1/2})$ and $\mathbf{w}_p$, $\mathbf{e}_e$, $\mathbf{w}_e$ of smaller orders of magnitude. All other combinations are mathematically possible also, but not all of them may have obvious physical relevance.

7. EULERIAN MEASURES

It is also of interest to construct approximate theories on the basis of finite Eulerian measures. For this purpose, we employ left polar decompositions

$$\mathbf{F} = \mathbf{N}\mathbf{R}, \quad \mathbf{F}_e = \mathbf{N}_e\mathbf{R}_e, \quad \mathbf{F}_p = \mathbf{N}_p\mathbf{R}_p \tag{7.1}$$

in place of (2.2). Here, $\mathbf{N}$, $\mathbf{N}_e$, and $\mathbf{N}_p$ are symmetric, positive definite left stretch tensors. We define Eulerian finite strain tensors by

$$\bar{\mathbf{E}} = \tfrac{1}{2}(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}), \quad \bar{\mathbf{E}}_e = \tfrac{1}{2}(\mathbf{I} - \mathbf{F}_e^{-T}\mathbf{F}_e^{-1}), \quad \bar{\mathbf{E}}_p = \tfrac{1}{2}(\mathbf{I} - \mathbf{F}_p^{-T}\mathbf{F}_p^{-1}). \tag{7.2}$$

where $\mathbf{F}^{-T} = (\mathbf{F}^{-1})^T = (\mathbf{F}^T)^{-1}$.

From (2.1) and $(7.2)_{1,2,3}$ it follows that

$$\bar{\mathbf{E}} - \bar{\mathbf{E}}_e = \mathbf{F}_e^{-T}\bar{\mathbf{E}}_p\mathbf{F}_e^{-1}. \tag{7.3}$$

Instead of $(2.5)_{1,2,3}$, we introduce the displacement gradient $\bar{\mathbf{H}}$ (relative to the present configuration $\boldsymbol{\kappa}$) and analogous tensors associated with $\mathbf{F}_e$ and $\mathbf{F}_p$. Thus,

$$\bar{\mathbf{H}} = \mathbf{I} - \mathbf{F}^{-1}, \quad \bar{\mathbf{H}}_e = \mathbf{I} - \mathbf{F}_e^{-1}, \quad \bar{\mathbf{H}}_p = \mathbf{I} - \mathbf{F}_p^{-1}. \tag{7.4}$$

We also decompose $\bar{\mathbf{H}}$, $\bar{\mathbf{H}}_e$, and $\bar{\mathbf{H}}_p$ into their symmetric and skew-symmetric parts:

$$\begin{aligned}
\bar{\mathbf{H}} &= \bar{\mathbf{e}} + \bar{\mathbf{w}}, & \bar{\mathbf{H}}_e &= \bar{\mathbf{e}}_e + \bar{\mathbf{w}}_e, & \bar{\mathbf{H}}_p &= \bar{\mathbf{e}}_p + \bar{\mathbf{w}}_p,\\
\bar{\mathbf{e}} &= \tfrac{1}{2}(\bar{\mathbf{H}} + \bar{\mathbf{H}}^T), & \bar{\mathbf{e}}_e &= \tfrac{1}{2}(\bar{\mathbf{H}}_e + \bar{\mathbf{H}}^T), & \bar{\mathbf{e}}_p &= \tfrac{1}{2}(\bar{\mathbf{H}}_p + \bar{\mathbf{H}}_p^T),\\
\bar{\mathbf{w}} &= \tfrac{1}{2}(\bar{\mathbf{H}} - \bar{\mathbf{H}}^T), & \bar{\mathbf{w}} &= \tfrac{1}{2}(\bar{\mathbf{H}}_e - \bar{\mathbf{H}}_e^T), & \bar{\mathbf{w}}_p &= \tfrac{1}{2}(\bar{\mathbf{H}}_p - \bar{\mathbf{H}}_p^T).
\end{aligned} \tag{7.5}$$

In view of (2.1), $(7.4)_{1,2,3}$ and $(7.5)_{4,5,6,7,8,9}$,

$$\bar{\mathbf{H}} = \bar{\mathbf{H}}_e + \bar{\mathbf{H}}_p - \bar{\mathbf{H}}_p\bar{\mathbf{H}}_e, \tag{7.6}$$

while

$$\begin{aligned}
\bar{\mathbf{e}} &= \bar{\mathbf{e}}_e + \bar{\mathbf{e}}_p - \tfrac{1}{2}\{\bar{\mathbf{H}}_p\bar{\mathbf{H}}_e + \bar{\mathbf{H}}_e^T\bar{\mathbf{H}}_p^T\} \\
&= \bar{\mathbf{e}}_e + \bar{\mathbf{e}}_p - \tfrac{1}{2}\{\bar{\mathbf{e}}_p\bar{\mathbf{e}}_e + \bar{\mathbf{e}}_p\bar{\mathbf{w}}_e + \bar{\mathbf{w}}_p\bar{\mathbf{e}}_e + \bar{\mathbf{w}}_p\bar{\mathbf{w}}_e + \bar{\mathbf{e}}_e\bar{\mathbf{e}}_p - \bar{\mathbf{e}}_e\bar{\mathbf{w}}_p - \bar{\mathbf{w}}_e\bar{\mathbf{e}}_p + \bar{\mathbf{w}}_e\bar{\mathbf{w}}_p\},
\end{aligned} \tag{7.7}$$

and

$$\begin{aligned}
\bar{\mathbf{w}} &= \bar{\mathbf{w}}_e + \bar{\mathbf{w}}_p - \tfrac{1}{2}\{\bar{\mathbf{H}}_p\bar{\mathbf{H}}_e - \bar{\mathbf{H}}_e^T\bar{\mathbf{H}}_p^T\} \\
&= \bar{\mathbf{w}}_e + \bar{\mathbf{w}}_p - \tfrac{1}{2}\{\bar{\mathbf{e}}_p\bar{\mathbf{e}}_e + \bar{\mathbf{e}}_p\bar{\mathbf{w}}_e + \bar{\mathbf{w}}_p\bar{\mathbf{e}}_p + \bar{\mathbf{w}}_p\bar{\mathbf{w}}_e - \bar{\mathbf{e}}_e\bar{\mathbf{e}}_p + \bar{\mathbf{e}}_e\bar{\mathbf{w}}_p + \bar{\mathbf{w}}_e\bar{\mathbf{e}}_p - \bar{\mathbf{w}}_e\bar{\mathbf{w}}_p\}.
\end{aligned} \tag{7.8}$$

Using all the relations in (7.2), (7.4) and (7.5), we conclude that

$$\begin{aligned}
\bar{\mathbf{E}} &= \bar{\mathbf{e}} - \tfrac{1}{2}\bar{\mathbf{H}}^T\bar{\mathbf{H}} &&= \bar{\mathbf{e}} - \tfrac{1}{2}(\bar{\mathbf{e}}^2 + \bar{\mathbf{e}}\bar{\mathbf{w}} - \bar{\mathbf{w}}\bar{\mathbf{e}} - \bar{\mathbf{w}}^2),\\
\bar{\mathbf{E}}_e &= \bar{\mathbf{e}}_e - \tfrac{1}{2}\bar{\mathbf{H}}_e^T\bar{\mathbf{H}}_e &&= \bar{\mathbf{e}}_e - \tfrac{1}{2}(\bar{\mathbf{e}}_e^2 + \bar{\mathbf{e}}_e\bar{\mathbf{w}}_e - \bar{\mathbf{w}}_e\bar{\mathbf{e}}_e - \bar{\mathbf{w}}_e^2),\\
\bar{\mathbf{E}}_p &= \bar{\mathbf{e}}_p - \tfrac{1}{2}\bar{\mathbf{H}}_p^T\bar{\mathbf{H}}_p &&= \bar{\mathbf{e}}_p - \tfrac{1}{2}(\bar{\mathbf{e}}_p^2 + \bar{\mathbf{e}}_p\bar{\mathbf{w}}_p - \bar{\mathbf{w}}_p\bar{\mathbf{e}}_p - \bar{\mathbf{w}}_p^2).
\end{aligned} \tag{7.9}$$

For the construction of infinitesimal plasticity and second-order plasticity, we would employ a measure of smallness

$$\delta = \max\{\sup_{\kappa} \| \bar{\mathbf{H}}_e \|, \quad \sup_{\bar{\kappa}} \| \bar{\mathbf{H}}_p \|\}. \tag{7.10}$$

We observe that, by virtue of (3.1), $(4.8)_{1,2,3}$ and $(7.4)_{1,2,3}$,

$$\begin{aligned} \bar{\mathbf{H}} &= \mathbf{H} - \mathbf{H}^2 + O(\epsilon^3), \\ \bar{\mathbf{H}}_e &= \mathbf{H}_e - \mathbf{H}_e^2 + O(\epsilon^3), \\ \bar{\mathbf{H}}_p &= \mathbf{H}_p - \mathbf{H}_p^2 + O(\epsilon^3). \end{aligned} \tag{7.11}$$

Also, with the aid of $(7.4)_{1,2,3}$ and (7.10) we deduce that

$$\begin{aligned} \mathbf{F} &= \mathbf{I} + \bar{\mathbf{H}} + \bar{\mathbf{H}}^2 + O(\delta^3), \\ \mathbf{F}_e &= \mathbf{I} + \bar{\mathbf{H}}_e + \bar{\mathbf{H}}_e^2 + O(\delta^3), \\ \mathbf{F}_p &= \mathbf{I} + \bar{\mathbf{H}}_p + \bar{\mathbf{H}}_p^2 + O(\delta^3), \end{aligned} \tag{7.12}$$

and hence

$$\begin{aligned} \mathbf{H} &= \bar{\mathbf{H}} + \bar{\mathbf{H}}^2 + O(\delta^3), \\ \mathbf{H}_e &= \bar{\mathbf{H}}_e + \bar{\mathbf{H}}_e^2 + O(\delta^3), \\ \mathbf{H}_p &= \bar{\mathbf{H}}_p + \bar{\mathbf{H}}_p^2 + O(\delta^3). \end{aligned} \tag{7.13}$$

The relations (7.11) and (7.13) allow approximate theories constructed from Eulerian measures to be related to those discussed in Sections 3, 4, 5, and 6 of the present paper.

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